Journal of Engineering Mathematics, Vol. 4, No. 3, July 1970
Wolters-Noordhoff Publishing–Groningen
Printed in the Netherlands

On the Dynamics of Strings

L. J. F. BROER

Department of Physics, Technicological University, Eindhoven, the Netherlands

(Received November 12, 1969)

SUMMARY
In order to elucidate some points in the linearised theory of waves on a string the exact equations for flexible strings have been investigated, both for inextensible and elastic strings. In the latter case the equations turn out to be hyperbolic and quasi-linear. It is shown that the characteristics connected with transverse waves are exceptional.

1. Introduction

The linearised theory of the motion of a string of constant length has been serving for a long time as an excellent introduction to the methods of mathematical physics. Nevertheless the conventional treatment of this subject contains some conceptual difficulties. The simplest solutions are those in which both ends of the string are kept in fixed positions. It is not usual (neither advisable) to point out to beginning students that the status of these solutions might be a bit doubtful as the corresponding solutions of the exact theory do not exist. Other problems are connected with the important notion of conservation laws. There are two equations of this type which are quadratic in the variables. One of these usually is interpreted as the energy equation. This requires the introduction of a potential energy density, a concept hardly compatible with the idea of a string of exactly constant length. The interpretation of the second law is not obvious in the linear theory.

In the present paper we will show how these points can be elucidated by first setting up an exact theory and postponing the linearisation to a later stage. In this respect the paper is analogous to a foregoing paper [1] on elastic bars. Both papers are mainly of a didactical character. The material contained in them is not treated in any textbooks (as far as I know). Even references to papers which might show the student how to solve these problems are lacking. Nevertheless it is quite conceivable that a good deal of the following treatment lies hidden in research papers of the last century. As a search for these papers would have been rather time consuming the author preferred to start from scratch, stating explicitly that he ignores whether any originality could be claimed for the contents of these two papers. The first problem mentioned can be tackled in two ways. We can consider a string of constant length, tied to some elastically yielding support or we can suppose that the string itself is elastic. The first method, which is the simplest, is treated in sections 2 and 3. The second method is probably more realistic and yields an illustration of the notion of exceptional characteristics of hyperbolic equations. It is treated in the ensuing sections.

2. Motion of a String of Constant Length

We consider a homogeneous flexible string of constant length. The mass per unit length is denoted by ρ. As in the bar problem we have to choose between fixed and moving coordinates. The advantage of the latter choice is even greater here, so we will use as independent variables t and the arc length s, which serves to identify a particular material point on the string. Dependent variables are the position of this point, $\boldsymbol{r}(s, t)$ and the stress. The condition of constant length entails that $\boldsymbol{r}_s$ is a unit vector:

$$\boldsymbol{r}_s \cdot \boldsymbol{r}_s = 1 . \tag{1}$$

Differentiation of this relation yields:

$$\boldsymbol{r}_s \cdot \boldsymbol{r}_{ss} = 0 \tag{2}$$

and

$$\boldsymbol{r}_s \cdot \boldsymbol{r}_{st} = 0\,. \tag{3}$$

The geometric, respectively kinematical, meaning of these useful equations will be obvious.

The kinetic energy density of the string is $\frac{1}{2}\rho \boldsymbol{r}_t \cdot \boldsymbol{r}_t$. There is no deformation energy. Taking into account the constraint (1) we therefore assume for the action integral the expression:

$$W = \int L\,dt = \int\int [\tfrac{1}{2}\rho\, \boldsymbol{r}_t \cdot \boldsymbol{r}_t - \tfrac{1}{2}\sigma(\boldsymbol{r}_s \cdot \boldsymbol{r}_s - 1)]\,ds\,dt\,. \tag{4}$$

Variation of σ then leads to (1). Variation of $\boldsymbol{r}$ gives:

$$\rho \boldsymbol{r}_{tt} = (\sigma \boldsymbol{r}_s)_s \tag{5}$$

as the equation of motion. The multiplier σ apparently is the stress, $\sigma \boldsymbol{r}_s$ the force vector in a cross section. Equation (1) and (5) determine the motion and the variation of σ along the string. An explicit relation for this variation can be found by writing out the right-hand side of (5), multiplication by $\boldsymbol{r}_s$ and use of (1) and (2). This yields:

$$\sigma_s = \rho \boldsymbol{r}_{tt} \cdot \boldsymbol{r}_s \tag{6}$$

which means that the stress gradient equals the tangential component of the inertial force. This obvious truth is an example of physical insight lost by linearisation in a too early stage.

We now turn to the conservation laws. As only the differentials $\boldsymbol{r}$, t and s occur in (4) there must be at least five of them according to *Noether's* theorem. The three relations associated with $\boldsymbol{r}$ are the equations (5) however. Proceeding in the usual way we find from the transformation $t \rightarrow t + \delta t$ the relation:

$$(\tfrac{1}{2}\rho \boldsymbol{r}_t \cdot \boldsymbol{r}_t)_t = (\sigma \boldsymbol{r}_s \cdot \boldsymbol{r}_t)_s\,. \tag{7}$$

As $\sigma \boldsymbol{r}_s \cdot \boldsymbol{r}_t$ is the work per unit time in a cross section this is the energy equation. An argument about potential energy is not necessary at this point as no such a quantity does occur.

In this same way one can work out the transformation $s \rightarrow s + \delta s$. This yield:

$$(\rho \boldsymbol{r}_s \cdot \boldsymbol{r}_t)_t = (\sigma + \tfrac{1}{2}\rho \boldsymbol{r}_t \cdot \boldsymbol{r}_t)_s\,. \tag{8}$$

Inspection of the left-hand side shows that (8) is a conservation law for the tangential component of the momentum. This also is a quantity which gets lost on linearisation. It is easy to verify, using (5), that (8) and (6) are equivalent.

Another set of conservation laws is found when we observe that (4) is invariant for rotation of the coordinate axes. In this way we obtain the relations for the moment of momentum in the form:

$$(\rho \boldsymbol{r}_t \times \boldsymbol{r})_t = (\sigma \boldsymbol{r}_s \times \boldsymbol{r})_s\,. \tag{9}$$

For the present purpose it is not necessary to consider any specified initial and boundary conditions. We only remark that the latter are in general non linear even for an ideally elastic support. The reason for this complication is that these conditions involve the true stress. For instance, when the string is in equilibrium at a stress σ_0 with one of its ends ($s = 0$) in the origin the boundary condition for elastic support is:

$$\sigma \boldsymbol{r}_s - \sigma_0 \boldsymbol{e} = \bar{\bar{\alpha}} \boldsymbol{r} \quad \text{for} \quad s = 0$$

where $\bar{\bar{\alpha}}$ is some constant symmetrical tensor and $\boldsymbol{e}$ the unit vector in the direction of the equilibrium position of the string. The actual handling of conditions of this type would be rather difficult.

3. Small Slope Approximation

For the sake of simplicity we now consider motion in the x–y plane only. The equations, written out in components, then read:

$$x_s^2 + y_s^2 = 1 \tag{10}$$

$$\rho x_{tt} = (\sigma x_s)_s \tag{11}$$

$$\rho y_{tt} = (\sigma y_s)_s \,. \tag{12}$$

Equation (10) can be satisfied by putting:

$$x_s = \cos\theta, \quad y_s = \sin\theta \,.$$

When the motion is such that the string remains nearly parallel to the x-axis θ can be considered as a small quantity, of order ε say. The function $x(s, t)$ then will be even, $y(s, t)$ odd in ε. Inspection of (11) and (12) shows that $\sigma(s, t)$ will be even too. In order to find the solution up to terms of order ε^2 we therefore put:

$$x = s + \varepsilon^2 \xi(s, t)$$

$$y = \varepsilon \eta(s, t)$$

$$\sigma = \sigma_0 + \varepsilon^2 \tau(s, t) \,.$$

Equation (12) now reduces to

$$\rho \eta_{tt} = \sigma_0 \eta_{ss} \tag{13}$$

which is the familiar linearised wave equation.
From (10) we obtain:

$$\xi_s = -\tfrac{1}{2}\eta_s^2 \tag{14}$$

and from (11) or (6):

$$\tau_s = \rho(\eta_{tt}\eta_s + \xi_{tt}) \,. \tag{15}$$

Once the appropriate solution of (13) has been found (14) and (15) can be integrated. The constants of integration have to be determined from the terms of order ε^2 in the boundary conditions. Special solutions of (13) are the simple waves $\eta = f(s \pm at)$, $a^2 = \sigma_0/\rho$. It is easily verified that τ_s will be zero and therefore σ a constant. This is even true for exact simple wave solutions. We will return to this property in section 5. We now turn to the relations (7) and (8). Both are even in ε, we therefore write down only the terms of order ε^2. The energy equation (7) then yields:

$$(\tfrac{1}{2}\rho\eta_t^2)_t = \sigma_0(\eta_s\eta_t + \xi_t)_s \,. \tag{16}$$

Now $\xi_{ts} = \xi_{st} = -(\tfrac{1}{2}\eta_s^2)_t$ by (14). Therefore we find:

$$\tfrac{1}{2}(\rho\eta_t^2 + \sigma_0\eta_s^2)_t = \sigma_0(\eta_s\eta_t)_s \tag{17}$$

which is the "energy equation" corresponding to (13) in the familiar elementary theory. The term $\tfrac{1}{2}\sigma_0\eta_s^2$ is the so-called potential contribution to the energy density. In our treatment it is the contribution of the motion in x-direction to the energy flux or work term, which just happens to be written as a derivative with respect to t.

Proceeding in the same manner with (8) we obtain:

$$\rho(\eta_s\eta_t + \xi_t)_t = \tau_s + (\tfrac{1}{2}\rho\eta_t^2)_s \,. \tag{18}$$

Using (15) and (13) this reduces to:

$$\rho(\eta_s\eta_t)_t = \tfrac{1}{2}\sigma_0(\eta_t^2 + \eta_s^2)_s \tag{19}$$

which is the second quadratic conservation law to (13), referred to in the introduction. It is equivalent to the conservation of tangential momentum. But again the density $\rho\eta_s\eta_t$ is not the true density of this quantity as a term has been transferred from left to right. The density of tangential momentum really is, up to the present order,

$$\rho(\eta_s\eta_t+\xi_t)$$

where ξ can be expressed in terms of η by means of (14).

4. Elastic String

We now assume our string to be elastic. This means that we now deal with a potential energy or stored energy of deformation.

We maintain the assumption that the string is completely flexible. The stored energy per unit mass therefore is taken to be a given function $E(l)$ of the specific length $l=ds/dm$ only. As $l=\rho^{-1}$ is no longer constant we have to decide whether s or m, the mass between a point on the string and a reference point fixed on the string, is taken as independent variable. We settle for m, as this turns out to be somewhat more simple. These considerations are closely analogous to the elastic bar problem [1].

We now have to set up equations for the dependent variables $\boldsymbol{r}(m, t)$. The Lagrangian is taken to be the difference between kinetic and stored energy:

$$L = \int dm\left[\tfrac{1}{2}\eta_t^2 - E(l)\right]. \tag{20}$$

In order to perform the variation we observe that:

$$l^2 = \left(\frac{ds}{dm}\right)^2 = \boldsymbol{r}_m\cdot\boldsymbol{r}_m\,, \quad l\,dl = \boldsymbol{r}_m\cdot d\boldsymbol{r}_m\,. \tag{21}$$

In this way we obtain for the equations of motion:

$$\boldsymbol{r}_{tt} = \left[\frac{1}{l}\,\frac{dE}{dl}\cdot\boldsymbol{r}_m\right]_m. \tag{22}$$

As $l^{-1}\boldsymbol{r}_m=\boldsymbol{r}_s$ is a unit vector $dE/dl=\sigma$ is the stress in a cross section. Formally (22) and (5) therefore are the same. The difference is that now σ is a given function of $l(\boldsymbol{r}_m)$ whereas in (5) σ has to be determined from (1) which is an (unimportant) identity in the present context.

As for conservation laws, variation of t yields:

$$\left[\tfrac{1}{2}r_t^2+E(l)\right]_t = \left[\frac{1}{l}\,\frac{dE}{dl}\,\boldsymbol{r}_m\cdot\boldsymbol{r}_t\right]_m \tag{23}$$

which is the energy equation. The expression within the brackets at the right-hand side is equal to $\sigma\boldsymbol{r}_s\cdot\boldsymbol{r}_t$, which is the by now familiar work term. From variation of m we obtain:

$$(\boldsymbol{r}_m\cdot\boldsymbol{r}_t)_t = \left[\tfrac{1}{2}r_t^2 + l\,\frac{dE}{dl} - E\right]_m. \tag{24}$$

As $E-l(dE/dl)$ is an enthalpy like quantity equation (24) represents, just like a similar expression in the theory of the bar, a form of Bernouilli's law. There is also conservation of angular momentum but it is not necessary for us to go into this question.

A more interesting aspect of the equations (22) is that they obviously are in the form of a set of quasi-linear hyperbolic equations. Therefore the classical theory of these equations as expounded e.g. by Jefferies and Taniuti [2] is applicable. It is easier to do this for first order equations. One therefore puts:

$$\boldsymbol{r}_m = \boldsymbol{l}, \quad \boldsymbol{r}_t = \boldsymbol{v}$$

and obtains the set of equations:

$$\boldsymbol{v}_t = \left[\frac{1}{l}\frac{dE}{dl}\boldsymbol{l}\right]_m \tag{25}$$

$$\boldsymbol{l}_t = \boldsymbol{v}_m\,. \tag{26}$$

It turns out that this set of equations has the characteristic velocities $\pm a$, $\pm c$ where:

$$a^2 = \frac{1}{l}\frac{dE}{dl} = \rho\sigma \tag{27}$$

$$c^2 = \frac{d^2E}{dl^2}\,. \tag{28}$$

Obviously c is the velocity of longitudinal waves just like those discussed in the bar problem. a corresponds to the transverse waves of the string. (We have here $a^2 = \rho\sigma$ instead of $\rho^{-1}\sigma$ because velocity here means mass over time instead of length over time).

From (27) and (28) we deduce:

$$c^2 - a^2 = l\frac{d}{dl}\left(\frac{1}{l}\frac{dE}{dl}\right) = l\frac{d}{dl}a^2\,. \tag{29}$$

This means that $c > a$ as a will increase upon stretching the string. The fact that a and c really are the characteristic speeds is physically plausible and can be proved directly by computing the roots of the characteristic equation. It is easier however to show that simple wave solutions with these speeds do exist. According to the general theory this is an indirect proof.

5. Simple Wave Solutions

As the set of equations (25) and (26) is hyperbolic and homogeneously linear in the derivatives it must have four sets of simple wave solutions. For these solutions one of the relations

$$\frac{\partial}{\partial t} = \pm c\frac{\partial}{\partial m} \quad \text{or} \quad \frac{\partial}{\partial t} = \pm a\frac{\partial}{\partial m} \tag{3;)}$$

must hold. These simple waves are not without interest, in this section we will look briefly into their properties.

The first set is rather trivial. Suppose that $\boldsymbol{v}$ and $\boldsymbol{l}$ have the same constant direction everywhere. Then (25) and (26) reduce to equations for the length of these vectors:

$$v_t = [a^2 l]_m = \left(a^2 + l\frac{da^2}{dl}\right)_{lm} = c^2 l_m \tag{31}$$

$$l_t = v_m \tag{32}$$

from which the desired result is obvious. Of course (31) and (32) are equivalent to the equations of the elastic bar. It was to be expected that our problem would admit purely longitudinal waves, including the non-simple wave solutions of (31) and (32). For the present purpose these solutions are of minor interest however as they have no counterpart in the theory as treated in sections 2 and 3, nor in the traditional linear theory.

We now look for simple wave solutions for which $\partial/\partial t = -a(\partial/\partial m)$ say. From (25) and (26) we obtain:

$$-a\boldsymbol{v}_m = [a^2\boldsymbol{l}]_m \tag{33}$$

$$-a\boldsymbol{l}_m = \boldsymbol{v}_m\,. \tag{34}$$

Eliminating $\boldsymbol{v}_m$ we find:

$$a^2\boldsymbol{l}_m = [a^2\boldsymbol{l}]_m$$

which can be true only when $a(l)$, and therefore l, is constant. Therefore, when these solutions exist they describe motions at constant length. In order to construct solutions of this kind we put, restricting ourselves to two-dimensional motion:

$$l_x = l\cos\theta,\quad l_y = l\sin\theta \tag{35}$$

where l is a constant and $\theta=\theta(t-am)$, an arbitrary function. The components of $\boldsymbol{v}$ then are easily found by integration of (33) or (34). Adjusting the constants of integration in such a way that $\boldsymbol{v}=0$ for $\theta=0$ (simple wave moving into an undisturbed region) we find:

$$v_x = al(1-\cos\theta),\quad v_y = -al\sin\theta\,. \tag{36}$$

Together with (35) this constitutes the required solution. As l is a constant the stress $\sigma(l)$ is constant too. Therefore these solutions also satisfy equations (1) and (5). It is easy to verify that in this case:

$$\boldsymbol{v}_t\cdot\boldsymbol{l}=0$$

which is equivalent to $\boldsymbol{r}_{tt}\cdot\boldsymbol{r}_s=0$ and therefore corresponds to $\sigma_s=0$ in (6). The energy equation (23) now reduces to:

$$(\tfrac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v})_t = \sigma(\boldsymbol{v}\cdot\boldsymbol{l})_m \tag{37}$$

which corresponds to (7).

We notice in passing that the energy density is:

$$\tfrac{1}{2}(v_x^2+v_y^2) = a^2 l^2(1-\cos\theta)\,.$$

Comparing this with (36) we see that the ratio of energy to momentum in x-direction equals al, which is the wave velocity in length over time. The present solution can be considered as a purely transverse motion. They differ from the longitudinal motions in two respects. In the first place a is a constant in any transverse simple wave. This is not true for c in longitudinal waves (exept when E is exactly a quadratic function of l). In other words, the a-characteristics are exceptional. In the second place there are no non-simple transverse waves, that is waves involving both a and $-a$ but not $\pm c$ characteristics. We will not investigate this in detail. Physically it means that, as in a non-simple wave σ is not a constant longitudinal waves will be exited by transverse waves. In the last section we will consider this effect in the small-slope approximation.

6. Small Slope Approximation for Ideal Elastic Strings

In this section we consider the equations for plane motion of the string:

$$\begin{aligned} x_{tt} &= [a^2(l)x_m]_m \\ y_{tt} &= [a^2(l)y_m]_m \end{aligned} \tag{38}$$

where

$$a^2 = \frac{1}{l}\frac{dE}{dl} \tag{39}$$

and

$$l^2 = x_m^2 + y_m^2\,.$$

It is possible to expand everything in terms of a small parameter ε. The resulting expressions are quite complicated. We will therefore apply some simplifying restrictions.

In the first place we assume the string to be ideally elastic. This means that E is a quadratic expression in l. When the stress-free specific length is L we can write:

$$E = \tfrac{1}{2}Y(l-L)^2 \tag{40}$$

where Y is a constant now. The stress then is:

$$\sigma = \frac{dE}{dl} = Y(l-L) \tag{41}$$

and the velocity of longitudinal waves (with respect to the mass) is given by

$$c^2 = Y\,.$$

In equilibrium the string is stretched to a length l_0. This requires a stress given by

$$\sigma_0 = Y(l_0 - L)\,.$$

In terms of these quantities we can rewrite (40) and (41) as:

$$E = E_0 + \sigma_0(l-l_0) + \tfrac{1}{2}Y(l-l_0)^2$$
$$\sigma = \sigma_0 + Y(l-l_0)\,.$$

For the transversal wave speed a we find by substitution in (39) or integration of (29):

$$a^2 = c^2 - (c^2 - a_0^2)\frac{l_9}{l} \tag{42}$$

where

$$a_0^2 = \frac{\sigma_0}{l_0} = Y\frac{l_0 - L}{l_0} = c^2 \cdot \frac{l_0 - L}{l_0}\,. \tag{43}$$

The ratio of the wave speeds equals the square root of the relative stretching. For a metal wire e.g. a/c therefore usually will be rather small. We now look for solutions up to the 2nd order:

$$x = l_0(m + \varepsilon\xi_1 + \varepsilon^2\xi_2)$$
$$y = l_0(\quad \varepsilon\eta_1 + \varepsilon^2\eta_2)\,.$$

Upon substitution in the preceding formulae we find for the first-order equations:

$$\xi_{1_{tt}} - c^2\xi_{1_{mm}} = 0 \tag{44}$$
$$\eta_{1_{tt}} - a_0^2\eta_{1_{mm}} = 0 \tag{45}$$

and for the second order:

$$\xi_{2_{tt}} - c^2\xi_{2_{mm}} = (c^2 - a_0^2)(\tfrac{1}{2}\eta_{1_m}^2)_m \tag{46}$$
$$\eta_{2_{tt}} - a_0^2\eta_{2_{mm}} = (c^2 - a_0^2)(\xi_{1_m}\eta_{1_m})_m\,. \tag{47}$$

The first set describe the first order free waves. The second set yields upon solution the second order forced waves generated by the free waves. Obvious a forced transverse wave exists only when both longitudinal and transverse waves are present. According to (46) a transverse wave induces longitudinal motion on its own. Therefore there are "transverse" solutions for which both ξ_1 and η_2 are zero. These solutions are analogous to those discussed in section 3. In this case there is no first-order term in l. We find:

$$l = l_0\{1 + \varepsilon^2(\xi_{2_m} + \tfrac{1}{2}\eta_{1_m}^2)\}\,. \tag{48}$$

We first look for solutions with $l = l_0$ and, therefore, constant stress. Combining (46) and (48) we obtain:

$$\xi_{2_{tt}} - a^2\xi_{2_{mm}} = 0 \tag{49}$$

which is of the same form as (45). Simple wave solutions of (45) and (49), depending on $m - at$ e.g., and satisfying the condition of constant length are easily found. These solutions correspond to those given by (35) and (36). According to the general theory of characteristics (38) ought to have simple wave solutions propagating with velocity a_0 into an unperturbed region to the right. These waves are not exceptional, that is they involve also the characteristic velocities $-a_0$ and $-c$, but not c. Small slope approximations to these solutions are constructed as follows: The solution of (45) is:

$$\eta = f(m - a_0 t) + g(m + a_0 t)$$

where f and g are zero for positive values of the argument. Therefore η vanishes to the right of the characteristic $m = a_0 t$.

Next we put this solution in the right-hand side of (46) and determine a particular solution by known methods, e.g. using the propagator.

This solution will be non-vanishing behind the characteristic $m = ct$. As $c > a$ the solution between these two characteristics will satisfy the homogeneous wave equation. Therefore it can be made to vanish by a suitable choice of the homogeneous solution to be added to the particular solution. At this stage there is no freedom left in choosing the solution. Therefore the backward facing c-wave cannot be removed.

Finally we put the solution of (45) and (46) into the energy equation and retain terms of order ε^2 only. It turns out that ξ_2 drops out in this order. We find:

$$\frac{\partial}{\partial t}\left[\tfrac{1}{2}\eta_t^2 + \frac{a_0^2}{2}\eta_m^2\right]_t = [a_0^2 \eta_t \eta_m]_m . \tag{50}$$

This corresponds exactly to the energy equation in the traditional linear theory. The term $\frac{1}{2}a^2\eta_m^2$ is a real potential energy now and not a flux term in disguise. Changing the variable from m to x introduces terms of higher order than ε^2. It therefore seems to be justified to tell students, worried by the usual treatment of this potential energy term, that this can be explained by considering the effect of stretch.

REFERENCES

[1] L. J. F. Broer, *J. Eng. Math.*, 4 (1970) 1–8.
[2] A. Jeffrey and T. Taniuti, *Non-linear wave propagation*, Academic Press, New York and London, 1964.

Journal of Engineering Math., Vol. 4 (1970) 195–202